MORELLON Philippe SF2940 Session 1 1.

References : Timo Koski : Lecture notes : Probability and Random processes at KTH

Allan Gut : An intermediate course in probability

Remark : Pay attention . One or two questions at the exam might be chosen from the Home works

1/ Let Ω be countable and set $\mathcal{A} := \{A \subset \Omega \, ; \, A \text{ is finite or } A^c \text{ is finite}\}$.

To show : $\mathcal{A}$ is an algebra

$\mathcal{A}$ is not a σ-algebra

* (a) $\emptyset \in \mathcal{A}$.

(b) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

(c) Let $(A_n)_{n\geq 1}$; $A_n \in \mathcal{A} \;\forall n \geq 1$, then $\bigcup_{n=1}^{N} A_n \in \mathcal{A} \;\forall N \geq 1$.

(a) Automatic : $\emptyset$ is finite

(b) $A \in \mathcal{A} \Leftrightarrow A$ is finite or A^c is finite $\underset{A = (A^c)^c}{\Leftrightarrow}$ $(A^c)^c$ is finite or A^c is finite $\Leftrightarrow A^c \in \mathcal{A}$.

(c.) By induction on N. $N=1$: $A_1 \in \mathcal{A}$: ok

$N \Rightarrow N+1$: $\bigcup_{n=1}^{N+1} A_n = \left(\bigcup_{n=1}^{N} A_n\right) \cup \left(A_{N+1}\right)$:

(1) $\bigcup_{n=1}^{N} A_n$ and A_{N+1} are finite

(2) $\bigcup_{n=1}^{N} A_n$ and A_{N+1} are infinite

(3) $\bigcup_{n=1}^{N} A_n$ is finite and A_{N+1} is infinite or $\bigcup_{n=1}^{N} A_n$ is infinite and A_{N+1} is finite

(1) $\bigcup_{n=1}^{N} A_n$ and A_{N+1} are finite $\Rightarrow \bigcup_{n=1}^{N+1} A_n$ is finite

(2) $\left(\bigcup_{n=1}^{N} A_n\right)^c$ is finite and $(A_{N+1})^c$ is finite . $\bigcup_{n=1}^{N+1} A_n \in \mathcal{A} \Leftrightarrow \bigcup_{n=1}^{N+1} A_n$ is finite or $\underbrace{\left(\bigcup_{n=1}^{N+1} A_n\right)^c}$ is finite

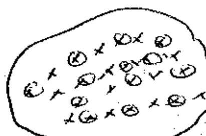
$= \underbrace{\left(\bigcup_{n=1}^{N} A_n\right)^c}_{\text{finite}} \cap \underbrace{A_{N+1}^c}_{\text{finite}} \Rightarrow$ ok

(3) $\left(\bigcup_{n=1}^{N} A_n\right)^c$ is infinite and A_{N+1}^c is finite $\Rightarrow \left(\bigcup_{n=1}^{N} A_n\right)^c \cap A_{N+1}^c$ is finite

since $\left(\bigcup_{n=1}^{N} A_n\right)^c \cap A_{N+1}^c \subset A_{N+1}^c$, which is finite

Identically, $\left(\bigcup_{n=1}^{N} A_n\right)^c$ is finite and A_{N+1}^c is infinite $\Rightarrow \left(\bigcup_{n=1}^{N} A_n\right)^c \cap A_{N+1}^c$ is finite

since $\left(\bigcup_{n=1}^{N} A_n\right)^c \cap A_{N+1}^c \subset \left(\bigcup_{n=1}^{N} A_n\right)^c$, which is finite



Ω

* Since Ω is countable, $\exists (x_i)_{i\geq 1}$; $x_i \in \Omega \;\forall i \geq 1$ and $x_i \neq x_j \;\forall i \neq j$.

Consider $(\{x_{2i}\})_{i\geq 1}$ and $(\{x_{2i+1}\})_{i\geq 1}$, two sequences of events .

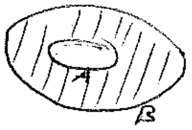
$\forall i \geq 1$, $\{x_{2i}\} \in \mathcal{A}$ since it is finite and $\{x_{2i+1}\} \in \mathcal{A}$ since it is finite

However, $\bigcup_{i\geq 1} \{x_{2i}\}$ is infinite and $\bigcup_{i\geq 1} \{x_{2i+1}\}$ is infinite and $\bigcup_{i\geq 1} \{x_{2i+1}\} \subseteq \left(\bigcup_{i\geq 1} \{x_{2i}\}\right)^c$

$\Rightarrow \left(\bigcup_{i\geq 1} \{x_{2i}\}\right)^c$ is infinite $\Rightarrow \bigcup_{i\geq 1} \{x_{2i}\} \notin \mathcal{A}$.

QED

[T.K] 1.12.2.6 : Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A, B \in \mathcal{F}$; $A \subseteq B$.
Show that $\mathbb{P}(B \cap A^c) = \mathbb{P}(B) - \mathbb{P}(A)$.



$$(B \cap A^c) \cap A = \emptyset$$

$$(B \cap A^c) \cup (B \cap A) = B \quad \text{and} \quad B \cap A = A \text{ since } A \subseteq B$$

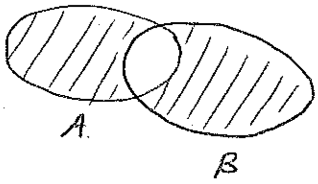
$$\Rightarrow B = (B \cap A^c) \sqcup A \quad \Rightarrow \mathbb{P}(B) = \mathbb{P}(B \cap A^c) + \mathbb{P}(A)$$

$$\Leftrightarrow \mathbb{P}(B) - \mathbb{P}(A) = \mathbb{P}(B \cap A^c)$$

QED

[T.K] 1.12.2.7 : Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $A, B \in \mathcal{F}$. Show that the probability that one and only one of the events A and B occurs is $\mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B)$.

Heuristic :



$$\{A \setminus (A \cap B)\} \sqcup \{B \setminus (A \cap B)\}$$

To make this formal, we are going to use the additivity of the measure

$$\mathbb{P}(\{A \setminus (A \cap B)\} \sqcup \{B \setminus (A \cap B)\}) = \mathbb{P}(A \setminus (A \cap B)) + \mathbb{P}(B \setminus (A \cap B))$$

$$\overset{1.12.2.6}{=} \mathbb{P}(A) - \mathbb{P}(A \cap B) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

$$= \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B)$$

QED

[T.K] 1.12.2.9. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Show that if $A, B \in \mathcal{F}$, then
$\min\{\mathbb{P}(A), \mathbb{P}(B)\} \geq \mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$.

Reminder : Let x, y, z be three real numbers. If $z \leq x$ and $z \leq y$, then $z \leq \min\{x, y\}$.

Here $A \cap B \subseteq A$ and $A \cap B \subseteq B$ $\overset{1.12.2.6}{\Rightarrow}$ $\mathbb{P}(A \cap B) \leq \mathbb{P}(A)$ and $\mathbb{P}(A \cap B) \leq \mathbb{P}(B)$ $\Rightarrow \mathbb{P}(A \cap B) \leq \min\{\mathbb{P}(A), \mathbb{P}(B)\}$

Now for the second inequality, we want to reexpress it so that it has a form that ressembles additivity.

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$$

$$\Leftrightarrow \mathbb{P}(A \cap B) \geq -\mathbb{P}(A^c) + 1 - 1 + \mathbb{P}(B)$$

$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

$$\Leftrightarrow \mathbb{P}(A \cap B) + \mathbb{P}(A^c) \geq \mathbb{P}(B).$$

Now we know that $\mathbb{P}(B) \overset{\text{add.}}{=} \mathbb{P}(B \cap A) + \mathbb{P}(B \cap A^c) \overset{1.12.2.6}{\leq} \mathbb{P}(B \cap A) + \mathbb{P}(A^c)$

$B \cap A^c \subseteq A^c$

which proves the result

QED

[T.K] 1.12.3.6. Markov's inequality : Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to \mathbb{R}$ a random variable. Suppose $\mathbb{P}(X \geq 0) = 1$. Show that $\forall c > 0$, $\mathbb{P}(X \geq c) \leq \frac{\mathbb{E}[X]}{c}$

$$X \geq c \mathbb{1}_{\{X \geq c\}} \Rightarrow \mathbb{E}[X] \geq \mathbb{E}[c \mathbb{1}_{\{X \geq c\}}] = c\,\mathbb{E}[\mathbb{1}_{\{X \geq c\}}] = c\,\mathbb{P}(X \geq c)$$

$$\Leftrightarrow \mathbb{P}(X \geq c) \leq \frac{\mathbb{E}[X]}{c}$$

course : $X \leq Y \Rightarrow \mathbb{E}[X] \leq \mathbb{E}[Y]$

QED

Remark : what can you say if $X \geq -M$ almost surely, $M > 0$?

[T.K] 1.2.3.10 Let $(\Omega, \mathcal{F}_1)$ and $(\Omega, \mathcal{F}_2)$ be two measurable spaces.

To show: $(\Omega, \mathcal{F}_1 \cap \mathcal{F}_2)$ is a measurable space.

We are going to show that the conditions that make $\mathcal{F}_1 \cap \mathcal{F}_2$ a σ-algebra are verified.

(1) $\emptyset \in \mathcal{F}_1 \cap \mathcal{F}_2$: OK since $\underbrace{\emptyset \in \mathcal{F}_1}_{\text{since } \mathcal{F}_1 \text{ is a } \sigma\text{-algebra}}$ and $\underbrace{\emptyset \in \mathcal{F}_2}_{\text{since } \mathcal{F}_2 \text{ is a } \sigma\text{-algebra}} \Rightarrow \emptyset \in \mathcal{F}_1 \cap \mathcal{F}_2$

(2.) $A \in \mathcal{F}_1 \cap \mathcal{F}_2 \Rightarrow A^c \in \mathcal{F}_1 \cap \mathcal{F}_2$: $A \in \mathcal{F}_1 \cap \mathcal{F}_2 \Rightarrow A \in \mathcal{F}_1$ and $A \in \mathcal{F}_2 \overset{\mathcal{F}_1, \mathcal{F}_2 \text{ are } \sigma\text{-algebras}}{\Rightarrow} A^c \in \mathcal{F}_1$ and $A^c \in \mathcal{F}_2 \Rightarrow A^c \in \mathcal{F}_1 \cap \mathcal{F}_2$: OK.

(3.) $A_n \in \mathcal{F}_1 \cap \mathcal{F}_2 \ \forall n \geq 1 \Rightarrow \bigcup_{n \geq 1} A_n \in \mathcal{F}_1 \cap \mathcal{F}_2$: $A_n \in \mathcal{F}_1 \cap \mathcal{F}_2 \ \forall n \geq 1 \Rightarrow A_n \in \mathcal{F}_1$ and $A_n \in \mathcal{F}_2 \ \forall n \geq 1 \underset{\mathcal{F}_1, \mathcal{F}_2 \text{ are } \sigma\text{-algebras.}}{\Rightarrow} \bigcup_{n \geq 1} A_n \in \mathcal{F}_1$ and $\bigcup_{n \geq 1} A_n \in \mathcal{F}_2$

$\Rightarrow \bigcup_{n \geq 1} A_n \in \mathcal{F}_1 \cap \mathcal{F}_2$: OK

$\Rightarrow \mathcal{F}_1 \cap \mathcal{F}_2$ is a σ-algebra $\Rightarrow$ $(\Omega, \mathcal{F}_1 \cap \mathcal{F}_2)$ is a measurable space.

QED.